MAT1322
Lecture 12

01/03/2017

TELESCOPIC SERIES, RATIO & ROOT TEST

<u>Ex</u> $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right)$

partial sums: $S_k = \sum_{n=1}^{k}\left(\frac{1}{n} - \frac{1}{n+1}\right)$

S_3: $\sum_{n=1}^{3}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \underbrace{\left(1-\tfrac{1}{2}\right)}_{n=1} + \underbrace{\left(\tfrac{1}{2} - \tfrac{1}{3}\right)}_{n=2} + \underbrace{\left(\tfrac{1}{3} - \tfrac{1}{4}\right)}_{n=3}$

$\boxed{= 1 - \tfrac{1}{4}}$

$S_k = \sum_{n=1}^{k}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1-\tfrac{1}{2}\right) + \left(\tfrac{1}{2} - \tfrac{1}{3}\right) + \left(\tfrac{1}{3} - \tfrac{1}{4}\right) + \ldots + \left(\frac{1}{k} - \frac{1}{k+1}\right)$

$\boxed{= 1 - \frac{1}{k+1}}$

REMEMBER: $\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} S_k = \lim_{k\to\infty} \sum_{n=1}^{k} a_n$

Here's a formula for the partial sums S_k. So we can compute the sum with $\lim_{k\to\infty} S_k$.

$\lim_{k\to\infty}\left(1 - \underbrace{\frac{1}{k+1}}_{\to 0}\right) \boxed{= 1}$ $\quad \therefore \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = 1$

<u>RECALL</u>

<u>absolute convergence</u>: A series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.

Absolute converge implies convergence BUT convergence does NOT imply absolute convergence.

<u>Ex</u> $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n}$ <u>converges</u> ($\ln(2)$)

but $\sum_{n=1}^{\infty} \frac{1}{n}$ <u>diverges</u>

TEST FOR CONVERGENCE (can check absolute convergence too)

RATIO TEST for $\sum_{n=1}^{\infty} a_n$

(R1) if $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L < 1$, then $\sum a_n$ converges absolutely

(R2) if $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L > 1$, then $\sum a_n$ diverges

If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 1$, then we can't make a conclusion.

Ex $\sum_{n=1}^{\infty} \frac{\sqrt{n}\cdot(-1)^n}{n^2+5}$

compute separately first

RATIO TEST $|a_{n+1}| = \frac{\sqrt{n+1}}{(n+1)^2+5} = \frac{\sqrt{n+1}}{n^2+2n+6}$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\sqrt{n+1}\cdot(n^2+5)}{\sqrt{n}\cdot(n^2+2n+6)}$$

... simplify ...

$= 1$ $\therefore$ no conclusion can be drawn.

ROOT TEST for $\sum_{n=1}^{\infty} a_n$

(1) if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$: absolute convergence

(2) if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$: diverges

If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$: no conclusion

geometric series $(q = \frac{2}{3})$

Ex $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{1-\frac{2}{3}} = \frac{1}{\frac{3-2}{3}} \boxed{= 3}$ $|q| < 1$, so it converges

sum $= \frac{1}{1-q} = \sum_{n=0}^{\infty} q^n$

ROOT TEST $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} (|a_n|)^{1/n}$

$$= \lim_{n\to\infty} \left(\left(\frac{2}{3}\right)^n\right)^{1/n} = \lim_{n\to\infty} \left(\frac{2}{3}\right)^{n\cdot\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{2}{3}\right) \boxed{= \frac{2}{3}} < 1$$

$\therefore \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$ converges absolutely, so it's also convergent.

Ex 2 $\sum_{n=1}^{\infty} \frac{1}{n^n}(-1)^n$. Check for absolute convergence.

n is in the exponent, so try the root test

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty}\left(\frac{1}{n^n}\right)^{1/n} = \lim_{n\to\infty}\left(\frac{1}{n^{n\cdot 1/n}}\right) = \lim_{n\to\infty}\left(\frac{1}{n}\right) = 0 < 1$$

$\therefore$ the series is absolutely convergent.

Ex 3 $\sum_{n=1}^{\infty}\left(-\frac{2n+3}{3n+2}\right)^n$ * use root test

* L'Hopital's

$$\lim_{n\to\infty}\sqrt[n]{\left(\frac{2n+3}{3n+2}\right)^n} = \lim_{n\to\infty}\left(\frac{2n+3}{3n+2}\right) = \lim_{n\to\infty}\left(\frac{2}{3}\right) = \frac{2}{3} < 1$$

$\therefore$ the series is absolutely convergent.

Comparison test example

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = \sum_{n=1}^{\infty}\frac{1}{n^2+n} \leq \sum_{n=1}^{\infty}\frac{1}{n^2}$$

This converges by the integral test because the exponent > 1

Integral test example

$\sum_{n=1}^{\infty} n^2 \cdot e^{-n^3}$ * the series converges iff the integral does

$$\int_1^{\infty} x^2 \cdot e^{-x^3}\,dx \quad \Bigg| \quad \begin{aligned} u &= x^3 \\ du &= 3x^2\,dx \end{aligned}$$

$$= \int_1^{\infty}\frac{1}{3}\cdot e^{-u}\,du$$

$$= \lim_{t\to\infty}\frac{1}{3}\cdot\int_1^{t^3} e^{-u}\,du = \frac{1}{3}\lim_{t\to\infty}\left(-e^{-u}\right)\Big|_1^{t^3}$$

$$= \frac{1}{3}\lim_{t\to\infty}\left(\underbrace{-e^{-t^3}}_{\to 0} - (-e^{-1})\right) = \frac{1}{3}e^{-1} = \boxed{\frac{1}{3e}}$$

Ratio test examples

1) $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$ $\qquad \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$ $\qquad$ if $L < 1$, abs. conv. if $L > 1$, div.

$|a_{n+1}| = \left(\frac{2}{3}\right)^{n+1}$

$$\lim_{n\to\infty} \frac{\left(\frac{2}{3}\right)^{n+1}}{\left(\frac{2}{3}\right)^n} = \lim_{n\to\infty} \frac{\cancel{\left(\frac{2}{3}\right)^n} \cdot \frac{2}{3}}{\cancel{\left(\frac{2}{3}\right)^n}} = \frac{2}{3} < 1, \quad \therefore \text{ converges (absolutely)}$$

2) $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ $\qquad a_{n+1} = \frac{n+1}{(n+1)^2+1} = \frac{n+1}{n^2+2n+2}$

$$\lim_{n\to\infty} \frac{(n+1)(n^2+1)}{n(n^2+2n+2)} = \lim_{n\to\infty} \frac{n^3+n+n^2+1}{n^3+2n^2+2n} = \lim_{n\to\infty} \frac{\frac{1}{n^3}(n^3+n+n^2+1)}{\frac{1}{n^3}(n^3+2n^2+2n)}$$

$$= \lim_{n\to\infty} \frac{1 + \overset{\to 0}{\frac{1}{n^2}} + \overset{\to 0}{\frac{1}{n^2}} + \overset{\to 0}{\frac{1}{n^3}}}{1 + \underbrace{\frac{2}{n} + \frac{2}{n^2}}_{\to 0}} = 1 \leftarrow \text{no conclusion!}$$

If there's no conclusion, we use a different test.

ie. the integral test

$$\int_1^{\infty} \frac{x}{x^2+1}\,dx \quad \Bigg| \quad \begin{aligned} u &= x^2+1 \\ du &= 2x\,dx \end{aligned}$$

$$= \lim_{t\to\infty} \int_2^{t^2+1} \frac{\cancel{x}}{u} \cdot \frac{du}{2\cancel{x}} = \frac{1}{2} \lim_{t\to\infty} \int_2^{t^2+1} \frac{1}{u}\,du$$

$$= \lim_{t\to\infty} \frac{1}{2} \cdot \ln(u)\Big]_2^{t^2+1} = \frac{1}{2} \lim_{t\to\infty} \left(\ln|t^2+1| - \ln(2)\right) \boxed{= \infty}$$

$\therefore$ the integral diverges, and so does the series.